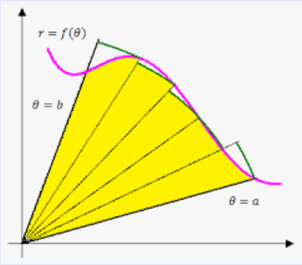
Calculus II

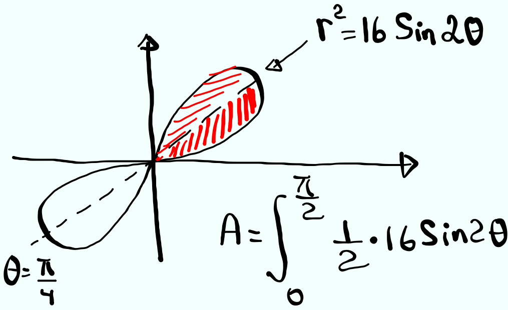
Lecture 19



Feb 19-8:47 AM

Class QZ 16

Use the formula $A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta$ to find the shaded area below.



$\theta = 0 \rightarrow r^2 = 0 \rightarrow r = 0$

$\theta = \frac{\pi}{4} \rightarrow r^2 = 16 \rightarrow r = 4$

$\theta = \frac{\pi}{2} \rightarrow r^2 = 0 \rightarrow r = 0$

$$A = \int_0^{\frac{\pi}{2}} \frac{1}{2} \cdot 16 \sin 2\theta \, d\theta = 8 \int_0^{\pi/2} \sin 2\theta \, d\theta$$

$$= 8 \cdot \frac{-\cos 2\theta}{2} \Big|_0^{\pi/2} = -4 \left[\cos \frac{2\pi}{2} - \cos 0 \right]$$

$$= -4 \left[\cos \pi - \cos 0 \right]$$

$$= -4 [-1 - 1] = \boxed{8}$$

Jul 15-6:59 AM

Sequence: It is a function whose domain is $\underbrace{\text{positive integers.}}_{1,\,2,\,3,\,4,\,\cdots,\,n,\,\cdots}$

Notation: $\underset{\substack{\uparrow\\ \text{First}\\ \text{Term}}}{a_1}, \underset{\substack{\uparrow\\ \text{Second}\\ \text{Term}}}{a_2}, a_3, \cdots, \underset{\substack{\uparrow\\ n\text{th term}}}{a_n}, \cdots$

$$\{a_n\} \qquad \{a_n\}_{n=1}^{\infty}$$

ex. $\{2n-1\}_{n=1}^{\infty}$

$$a_1 = 2(1) - 1 = 1$$

$$a_2 = 2(2) - 1 = 3$$

$$a_5 = 2(5) - 1 = 9$$

$$\vdots \qquad \vdots$$

$$a_{100} = 2(100) - 1 = 199$$

$$\vdots \qquad \vdots$$

Jul 15-8:16 AM

Write the $\underbrace{\text{first 3 terms}}$ of

$$\left\{\cos \frac{n\pi}{2}\right\}_{n=1}^{\infty} \longrightarrow 0,\ -1,\ 0$$

$$a_1 = \cos \frac{1\pi}{2} = 0$$

$$a_2 = \cos \frac{2\pi}{2} = \cos \pi = -1$$

$$a_3 = \cos \frac{3\pi}{2} = 0$$

Write the first 4 terms of $a_n = \frac{1+3^n}{2^n}$

$$a_1 = \frac{1+3^1}{2^1} = \frac{4}{2} = 2$$

$$a_2 = \frac{1+3^2}{2^2} = \frac{10}{4} = 2.5$$

$$a_3 = \frac{1+3^3}{2^3} = \frac{28}{8} = \frac{7}{2} = 3.5$$

$$a_4 = \frac{1+3^4}{2^4} = \frac{82}{16} = \frac{41}{8} = 5.125$$

Jul 15-8:20 AM

$a_1 = 3 \quad , \quad a_{n+1} = \frac{a_n}{n+1}$

$n=1$

$a_2 = \frac{a_1}{2} = \frac{3}{2}$

$n=2$

$a_3 = \frac{a_2}{3} = \frac{\frac{3}{2}}{3} = \frac{1}{2}$

$n=3$

$a_4 = \frac{a_3}{4} = \frac{\frac{1}{2}}{4} = \frac{1}{8}$

Jul 15-8:26 AM

Find the next 3 terms when

$a_1 = 2 \quad , \quad a_{n+1} = \frac{a_n}{1+a_n}$

$n=1$ $\quad a_2 = \frac{a_1}{1+a_1} = \frac{2}{3}$

$n=2$ $\quad a_3 = \frac{a_2}{1+a_2} = \frac{\frac{2}{3}}{1+\frac{2}{3}} = \frac{2}{5}$

$n=3$ $\quad a_4 = \frac{a_3}{1+a_3} = \frac{\frac{2}{5}}{1+\frac{2}{5}} = \frac{2}{7}$

Jul 15-8:29 AM

Write the first 5 terms of the sequence given by $a_1 = 1$, $a_2 = 1$, $a_{n+2} = a_{n+1} + a_n$

$n=1 \rightarrow a_3 = a_2 + a_1 = 1 + 1 = 2$

$n=2 \rightarrow a_4 = a_3 + a_2 = 2+1 = 3$

$n=3 \rightarrow a_5 = a_4 + a_3 = 3+2=5$

$n=4 \rightarrow a_6 = a_5 + a_4 = 5+3=8$

$1, 1, 2, 3, 5, 8, 13, 21, \ldots$

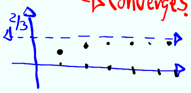
Fibonacci Sequence

If $\lim_{n\to\infty} a_n = L$ and L is defined, then the sequence $\{a_n\}$ Converges, otherwise diverges.

$\left\{\frac{2n-1}{3n+1}\right\}$ → Converges.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n-1}{3n+1} = \frac{\infty}{\infty}$$

$$= \lim_{n\to\infty} \frac{2}{3} = \underset{L}{\boxed{\frac{2}{3}}}$$



Jul 15-8:33 AM

$$a_n = \sqrt{\frac{n+1}{9n+1}}$$

1) Give the first 3 terms.

$$a_1 = \sqrt{\frac{2}{10}} = \sqrt{\frac{1}{5}} = \frac{\sqrt{5}}{5} \qquad a_2 = \sqrt{\frac{3}{19}} = \frac{\sqrt{57}}{19}$$

$$a_3 = \sqrt{\frac{4}{28}} = \sqrt{\frac{1}{7}} = \frac{\sqrt{7}}{7}$$

2) $\lim_{n\to\infty} a_n$

$$= \lim_{n\to\infty} \sqrt{\frac{n+1}{9n+1}} = \sqrt{\lim_{n\to\infty} \frac{n+1}{9n+1}} = \sqrt{\frac{1}{9}} = \frac{1}{3}$$

3) Does $\{a_n\}$ $\boxed{\text{converge}}$ or diverge? explain

$\lim_{n\to\infty} a_n = \frac{1}{3}$ defined

Jul 15-8:44 AM

Given $\left\{\frac{n!}{(n+2)!}\right\}$

$n! = n(n-1)(n-2)\cdots3\cdot2\cdot1$

$4! = 4\cdot3\cdot2\cdot1 = 24$

$10! = 10\cdot9\cdot8\cdots3\cdot2\cdot1$

$1! = 1,\ 0! = 1$

Converges

$a_1 = \frac{1!}{3!} = \frac{1}{3\cdot2\cdot1} = \frac{1}{6}$

$a_2 = \frac{2!}{4!} = \frac{\cancel{2\cdot1}}{4\cdot3\cdot\cancel{2\cdot1}} = \frac{1}{12}$

$a_3 = \frac{3!}{5!} = \frac{\cancel{3\cdot2\cdot1}}{5\cdot4\cdot\cancel{3\cdot2\cdot1}} = \frac{1}{20}$

$a_4 = \frac{4!}{6!} = \frac{\cancel{4!}}{6\cdot5\cdot\cancel{4!}} = \frac{1}{30}$

Now $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\frac{n!}{(n+2)!} = \lim\limits_{n\to\infty}\frac{\cancel{n!}}{(n+2)\cdot(n+1)\cdot\cancel{n!}}$

$= \lim\limits_{n\to\infty}\frac{1}{(n+2)(n+1)} = \boxed{0}$

Jul 15-8:53 AM

Given $a_n = \ln(n+1) - \ln n$

1) Write the first 4 terms.

$a_1 = \ln 2 - \ln 1 = \ln 2$

$a_2 = \ln 3 - \ln 2 = \ln\frac{3}{2}$

$a_3 = \ln 4 - \ln 3 = \ln\frac{4}{3}$

$a_4 = \ln 5 - \ln 4 = \ln\frac{5}{4}$

2) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\left[\ln(n+1) - \ln n\right] = \infty - \infty$

$= \lim\limits_{n\to\infty} \ln\frac{n+1}{n} = \ln \lim\limits_{n\to\infty}\frac{n+1}{n} = \ln 1 = \boxed{0}$

3) Determine whether a_n converges or diverges. Explain. Converges, $\lim\limits_{n\to\infty} a_n$ is defined

Jul 15-9:01 AM

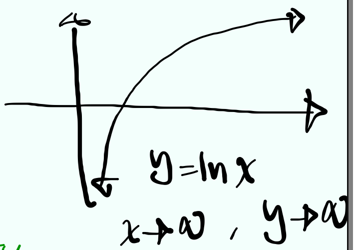
Determine whether the sequence $a_n = \tan^{-1}(\ln n)$ converge or diverge.

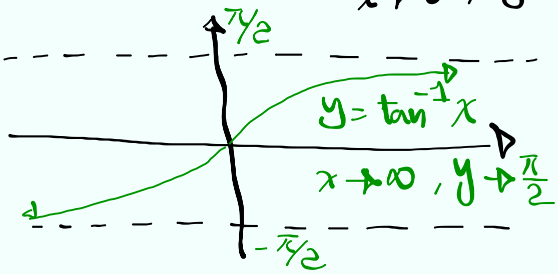
$a_1 = \tan^{-1}(\ln 1) = \tan^{-1} 0 = 0$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \tan^{-1}(\overset{\infty}{\cancel{\ln n}})$$

$$= \frac{\pi}{2}$$

Converges.





Jul 15-9:09 AM

If $a_n < a_{n+1}$ → sequence a_n is Increasing.

If $a_n > a_{n+1}$ → sequence a_n is decreasing.

Sequence a_n is monotonic if it is increasing or decreasing.

Determine $\left\{\frac{2n-1}{3n+1}\right\}$ whether increasing or decreasing.

$a_n = \frac{2n-1}{3n+1}$ $\qquad a_{n+1} = \frac{2(n+1)-1}{3(n+1)+1}$

$a_{n+1} = \frac{2n+1}{3n+4}$

Assume $a_n > a_{n+1}$

$$\frac{2n-1}{3n+1} > \frac{2n+1}{3n+4}$$

Cross-Multiply

$(2n-1)(3n+4) > (3n+1)(2n+1)$

$\cancel{6n^2} + \cancel{5n} - 4 > \cancel{6n^2} + \cancel{5n} + 1$

$-4 > 1$

False

So $a_n < a_{n+1}$

Sequence is increasing.

Jul 15-9:14 AM

Use Calc I to determine whether

$\left\{\frac{2n-1}{3n+1}\right\}$ $\boxed{\text{increasing}}$ or decreasing.

$f(x)=\frac{2x-1}{3x+1}$ $x\geq 1$

$$f'(x)=\frac{2(3x+1)-(2x-1)\cdot 3}{(3x+1)^2}$$

$$=\frac{6x+2-6x+3}{(3x+1)^2}=\frac{5}{(3x+1)^2}>0$$

Jul 15-9:22 AM

$a_n=ne^{-n}$, Is it increasing or decreasing?

$a_n=\frac{n}{e^n}$

$a_1=\frac{1}{e}$

$a_2=\frac{2}{e^2}$

$a_3=\frac{3}{e^3}$

$\boxed{\text{Decreasing}}$

$a_n>a_{n+1}$

$\frac{n}{e^n}>\frac{n+1}{e^{n+1}}$

$\frac{n}{\cancel{e^n}}>\frac{n+1}{\cancel{e^n}\cdot e^1}$

$n>\frac{n+1}{e}$

$en>n+1$

True ✓

$f(x)=\frac{x}{e^x}$

$f'(x)=\frac{1\cdot e^x-x\cdot e^x}{(e^x)^2}$

$=\frac{e^x(1-x)}{e^{2x}}=\frac{1-x}{e^x}$

$e^x>0$, $1-x\leq 0$ for $x\geq 1$ $f'(x)<0$

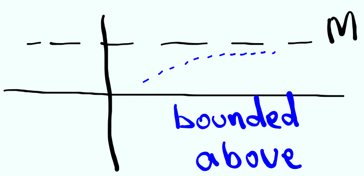
Jul 15-9:26 AM

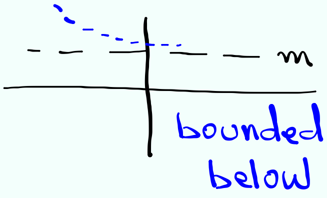
If $a_n \le M$ for all $n \ge 1$,

then $\{a_n\}$ is bounded above.

If $m \le a_n$ for all $n \ge 1$,

then $\{a_n\}$ is bounded below.





Theorem: Every bounded monotonic sequence is convergent.

Jul 15-9:34 AM

Given $a_1 = 2$, $a_{n+1} = \frac{1}{2}(a_n + 6)$

1) write the first 4 terms

$a_2 = \frac{1}{2}(a_1+6) = \frac{1}{2}(2+6) = \frac{1}{2}(8) = 4$

$a_3 = \frac{1}{2}(a_2+6) = \frac{1}{2}(4+6) = \frac{1}{2}(10) = 5$

$a_4 = \frac{1}{2}(a_3+6) = \frac{1}{2}(5+6) = \frac{1}{2}(11) = \frac{11}{2} = 5.5$

2, 4, 5, 5.5 Increasing

show that $a_n < 6$ bounded above by 6.

$a_n + 6 < 6 + 6$

$a_n + 6 < 12$

$a_{n+1} = \boxed{\frac{1}{2}}(a_n+6)$

$\boxed{\frac{1}{2}(a_n+6)} < \frac{1}{2} \cdot 12$

$a_{n+1} < 6$

by Mathematical Induction $a_n < 6$

Increasing, bounded → $\{a_n\}$ Converges.

Jul 15-9:40 AM

Show $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

Sum $1 + 2 + 3 + \cdots + 100 = \frac{100(100+1)}{2}$

$= 50(101)$

$= \boxed{5050}$

M.I.: Is it true for $n=1$?

$1 = \frac{1(1+1)}{2} \Rightarrow 1 = \frac{1\cdot 2}{2} = 1$ ✓

Is it true for $n=2$?

$1 + 2 = \frac{2(2+1)}{2} = 3$ ✓

what about $n=3$?

$1 + 2 + 3 = \frac{3(3+1)}{2}$

$6 = \frac{3\cdot 4}{2} = 6$ ✓

Jul 15-9:49 AM

Assume it works for $n=K$

$1 + 2 + 3 + \cdots + K + \boxed{K+1} = \frac{K(K+1)}{2} + K+1$

$= \frac{K(K+1)}{2} + \frac{2(K+1)}{2}$

$= \frac{K(K+1) + 2(K+1)}{2}$

$= \frac{(K+1)(K+2)}{2}$

$= \frac{(K+1)(K+1+1)}{2}$

It works by M.I.

Jul 15-9:54 AM

Show $1^2+2^2+3^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$

ex: $n=3$

$$1^2+2^2+3^2=\frac{3(3+1)(2\cdot3+1)}{6}$$

$$1+4+9=\frac{\cancel{3}\cdot\cancel{4}\cdot7}{\cancel{6}}$$

$$14=2\cdot7$$

Assume it works for $n=K$

$$1^2+2^2+3^2+\cdots+K^2+\boxed{(K+1)^2}=\frac{K(K+1)(2K+1)}{6}+(K+1)^2$$

RHS $\frac{K(K+1)(2K+1)}{6}+\frac{6(K+1)^2}{6}$

$$=\frac{(K+1)[K(2K+1)+6(K+1)]}{6}$$

$$=\frac{(K+1)[2K^2+7K+6]}{6}=\frac{(K+1)[(K+2)(2K+3)]}{6}$$

Final Ans $=\frac{(K+1)(K+1+1)(2(K+1)+1)}{6}$

we proved $1^2+2^2+3^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$

by M.I.

Jul 15-9:58 AM